Symmetries at Causal Boundaries

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Symmetry

Theory: An action with some boundary/initial conditions.

Solution space: All solutions of the theory compatible with the boundary/initial conditions.

Symmetry: Transformations which rotate us in solution space

$$\text{solution}_1 \xrightarrow{\text{symmetry}} \text{solution}_2$$

different solutions correspond to different boundary data

$$\text{boundary data}_1 \xrightarrow{\text{symmetry}} \text{boundary data}_2$$

Conserved charge: There exists conserved quantities associated with these symmetries. (Noether's theorem)

Example, point particle

Action: $S = \int \frac{1}{2} m\dot{q}^2 \, dt$

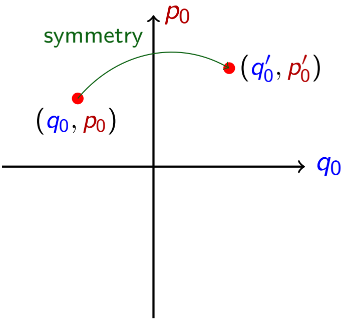
Equation of motion: $m\ddot{q} = 0$

Solutions: $q(t) = \frac{p_0}{m} t + q_0$

Initial data: $\{q_0, p_0\}$

Trans. symmetry: $\delta_\epsilon q_0 = \epsilon$, $\delta_\epsilon p_0 = 0$

Conserved charge: Momentum



Solution phase space

Einstein's gravity action:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^3x\,\sqrt{-g}\,(R-2\Lambda)\,, \qquad \mathcal{E}_{\mu\nu} := R_{\mu\nu} - 2\Lambda g_{\mu\nu} = 0$$

Metric ansatz:

$$ds^2 = -Vdv^2 + 2\eta dvdr + \mathcal{R}^2\left(d\phi + Udv\right)^2$$

Equations of motion result:

$$\begin{aligned}
\mathcal{R} =& \Omega + \lambda\,\eta\,r \\
U =& \mathcal{U} + \frac{1}{\lambda\mathcal{R}}\,\frac{\partial_\phi\eta}{\eta} + \frac{\Upsilon}{2\lambda\mathcal{R}^2} \\
V =& \frac{1}{\lambda^2}\left(-\Lambda\mathcal{R}^2 - \mathcal{M} + \frac{\Upsilon^2}{4\mathcal{R}^2} - \frac{2\mathcal{R}}{\eta}\mathcal{D}_v(\eta\lambda) + \frac{\Upsilon}{\mathcal{R}}\,\frac{\partial_\phi\eta}{\eta}\right)
\end{aligned}$$

Solution phase space

Two constraint equations:

$$\mathcal{E}_{\hat{\mathcal{M}}} := \mathcal{D}_v \hat{\mathcal{M}} + \Lambda\lambda\partial_\phi\left(\frac{\hat{\Upsilon}}{\lambda^2}\right) + 2\partial_\phi^3\mathcal{U} = 0$$

$$\mathcal{E}_{\hat{\Upsilon}} := \mathcal{D}_v \hat{\Upsilon} - \lambda\partial_\phi\left(\frac{\hat{\mathcal{M}}}{\lambda^2}\right) + 2\partial_\phi^3(\lambda^{-1}) = 0$$

Comoving derivative:

$$\begin{aligned} \mathcal{D}_v O_w &:= \partial_v O_w - \mathcal{L}_{\mathcal{U}} O_w \\ \mathcal{L}_{\mathcal{U}} O_w &:= \mathcal{U}\partial_\phi O_w + w O_w \partial_\phi \mathcal{U} \end{aligned}$$

Boundary data: $\{\eta(v,\phi), \Omega(v,\phi), \hat{\Upsilon}(v,\phi), \hat{\mathcal{M}}(v,\phi)\}$

Causal boundary symmetry

Causal boundary symmetry:

$$\xi = T\partial_v + \left[Z - \frac{r}{2} W - \frac{\Upsilon \partial_\phi T}{2\eta\lambda^2\mathcal{R}} - \frac{1}{\eta^2\lambda}\partial_\phi \left(\frac{\eta\partial_\phi T}{\lambda} \right) \right] \partial_r + \left(Y + \frac{\partial_\phi T}{\lambda\mathcal{R}} \right) \partial_\phi$$

$T(v,\phi)$: supertranslation in v-direction

$Z(v,\phi)$: supertranslation in r-direction

$W(v,\phi)$: superscaling in r-direction

$Y(v,\phi)$: superrotation in ϕ-direction

Transformations law

Transformations law:

$$\{\eta, \Omega, \hat{\Upsilon}, \hat{\mathcal{M}}\} \xrightarrow{\ \xi\ } \{\eta + \delta_\xi \eta, \Omega + \delta_\xi \Omega, \hat{\Upsilon} + \delta_\xi \hat{\Upsilon}, \hat{\mathcal{M}} + \delta_\xi \hat{\mathcal{M}}\}$$

Explicit forms:

$$\delta_\xi \eta = \mathcal{D}_v(T\eta) + \hat{Y}\partial_\phi \eta - \frac{1}{2}\eta W$$

$$\delta_\xi \Omega = T\,\mathcal{D}_v \Omega + \partial_\phi(\Omega \hat{Y}) + \eta\lambda Z$$

$$\delta_\xi \hat{\Upsilon} \approx \hat{T}\partial_\phi \hat{\mathcal{M}} + 2\hat{\mathcal{M}}\partial_\phi \hat{T} + \hat{Y}\partial_\phi \hat{\Upsilon} + 2\hat{\Upsilon}\partial_\phi \hat{Y} - 2\partial_\phi^3 \hat{T}$$

$$\delta_\xi \hat{\mathcal{M}} \approx \hat{Y}\partial_\phi \hat{\mathcal{M}} + 2\hat{\mathcal{M}}\partial_\phi \hat{Y} - \Lambda(\hat{T}\partial_\phi \hat{\Upsilon} + 2\hat{\Upsilon}\partial_\phi \hat{T}) - 2\partial_\phi^3 \hat{Y}$$

where $\hat{Y} = Y + \mathcal{U}T$ and $\hat{T} = T/\lambda$.

Causal boundary symmetry algebra

Causal boundary symmetry algebra:

$$[\xi(T_1, Z_1, W_1, Y_1), \xi(T_2, Z_2, W_2, Y_2)]_{\text{adj. bracket}} = \xi(T_{12}, Z_{12}, W_{12}, Y_{12})$$

with

$$\begin{aligned}
T_{12} &= (T_1\partial_v + Y_1\partial_\phi)T_2 - (1 \leftrightarrow 2)\\
Z_{12} &= (T_1\partial_v + Y_1\partial_\phi)Z_2 + \frac{1}{2}W_1 Z_2 - (1 \leftrightarrow 2)\\
W_{12} &= (T_1\partial_v + Y_1\partial_\phi)W_2 - (1 \leftrightarrow 2)\\
Y_{12} &= (T_1\partial_v + Y_1\partial_\phi)Y_2 - (1 \leftrightarrow 2)
\end{aligned}$$

Causal boundary symmetry algebra: $\text{Diff}(\mathcal{C}) \in \text{Weyl}(\mathcal{C}) \in \text{T}_r(\mathcal{C})$

Surface charge analysis

Surface charge:

$$\delta\!\!\!/ Q_\xi = \frac{1}{16\pi G}\oint_{\mathcal{C}_{r,v}} d\phi\left[W\delta\Omega + 2Z\delta(\Omega\, e^{\Pi/2}) + Y\delta\Upsilon + T\delta\!\!\!/\mathcal{H}\right].$$

$$\delta\!\!\!/\mathcal{H} := -\mathcal{D}_v\Pi\,\delta\Omega + \mathcal{U}\delta\Upsilon + \mathcal{D}_v\Omega\delta\Pi + \lambda^{-1}\delta\hat{\mathcal{M}}, \qquad \Pi := \ln\left(\frac{\eta\lambda}{\Omega}\right)^2$$

here we have added the following Y-term

$$Y^{\mu\nu}[g;\delta g] = \frac{\delta\sqrt{-g}}{16\pi G}\,\epsilon^{\mu\nu}\,.$$

Our symplectic quantities are radial independent.

Heisenberg slicing

Change of slicing:

$$\hat{Z} = \delta_\xi \Omega\,, \qquad \hat{W} = -\delta_\xi \Pi\,, \qquad \hat{Y} = Y + \mathcal{U}\, T\,, \qquad \hat{T} = \frac{T}{\lambda}\,.$$

Charge in Heisenberg slicing:

$$\delta Q_\xi = \frac{1}{16\pi G} \oint_{\mathcal{C}_{r,v}} d\phi \left(\hat{W} \delta\Omega + \hat{Z} \delta\Pi + \hat{Y} \delta\hat{\Upsilon} + \hat{T} \delta\hat{\mathcal{M}} \right).$$

If we assume $\delta \hat{Z} = \delta \hat{Y} = \delta \hat{W} = \delta \hat{T} = 0\,,$

$$Q_\xi = \frac{1}{16\pi G} \oint_{\mathcal{C}_{r,v}} d\phi \left(\hat{W} \Omega + \hat{Z} \Pi + \hat{Y} \hat{\Upsilon} + \hat{T} \hat{\mathcal{M}} \right).$$

Surface charge algebra

Charge algebra in Heisenberg slicing:

$$\{\Omega(v,\phi), \Pi(v,\phi')\} = 16\pi G\ \delta\left(\phi-\phi'\right)$$

$$\{\hat{\Upsilon}(v,\phi), \hat{\Upsilon}(v,\phi')\} = 16\pi G \left(\hat{\Upsilon}(v,\phi')\partial_\phi - \hat{\Upsilon}(v,\phi)\partial_{\phi'}\right)\delta\left(\phi-\phi'\right)$$

$$\{\hat{\mathcal{M}}(v,\phi), \hat{\mathcal{M}}(v,\phi')\} = -16\pi G\Lambda \left(\hat{\Upsilon}(v,\phi')\partial_\phi - \hat{\Upsilon}(v,\phi)\partial_{\phi'}\right)\delta\left(\phi-\phi'\right)$$

$$\{\hat{\Upsilon}(v,\phi), \hat{\mathcal{M}}(v,\phi')\} = 16\pi G \left(\hat{\mathcal{M}}(v,\phi')\partial_\phi - \hat{\mathcal{M}}(v,\phi)\partial_{\phi'} - 2\partial_\phi^3\right)\delta\left(\phi-\phi'\right)$$

Discussion and concluding remarks

- Quantization.
- Generalization to higher dimensions.
- Well-defined action principle and boundary theory.
- Local thermodynamic descriptions. [See Dr. Sheikh-Jabbari's talk]

Thank you for your attention!